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Letter to the Editor

A Short Proof of Müntz's Theorem

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Let P be the set of all finite linear combinations of the functions $1, x^{\lambda_1}, x^{\lambda_2},\ldots,$ where $0 < \lambda_k \to \infty$. Müntz's second theorem states that P is dense in $C[0, 1]$ (in the uniform norm $\|\cdot\|$) if and only if $\sum(1/\lambda_k) = \infty$.

It is a classical idea to use the Weierstrass approximation theorem to reduce the proof to that of the approximability of the monomials x^q. There are many proofs of the inclusion of x^q in the (uniform norm) closure of P, but they either introduce the L^2 norm or involve more complicated arguments in the uniform norm. We provide an elementary constructive proof.

Let $q > 0$, $q \neq \lambda_k$. Define functions Q_n of the form

$$Q_n(x) = x^q - \sum_{k=1}^{n} a_{kn} x^{\lambda_k}, \qquad 0 < x \leqslant 1, \tag{1}$$

by the recurrence $Q_0(x) = x^q$,

$$Q_n(x) = (\lambda_n - q)\, x^{\lambda_n} \int_x^1 Q_{n-1}(t)\, t^{-1-\lambda_n}\, dt, \qquad n = 1, 2,\ldots.$$

Since $\|Q_0\| = 1$ and $\|Q_n\| \leqslant |1 - (q/\lambda_n)|\, \|Q_{n-1}\|$,

$$\|Q_n\| \leqslant \prod_{k=1}^{n} |1 - (q/\lambda_k)|. \tag{2}$$

Hence, $\|Q_n\| \to 0$ if $\sum(1/\lambda_k) = \infty$.

For example, by taking $q = 1$, $\lambda_k = 2k$, one obtains another *constructive* proof of the approximability of $|x|$ by polynomials on $[-1, 1]$. But we have to admit that the upper bound (2) can be improved by turning to linear

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combinations of best approximation. Namely, using known results for the L^2 norm, the author (*J. Approx. Theory* **3** (1970), 72–86) has proved that

$$\min_{c_k} \left\| x^q - \sum_{k=1}^{n} c_k x^{\lambda_k} \right\| \leqslant \prod_{k=1}^{n} \frac{|\lambda_k - q|}{\lambda_k + q}.$$